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Counting and Arithmetic of the Inca

Cuentas y Aritmética de los Inca

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Abstract

The Inca Empire - the greatest pre-Columbian empire on the American continent - extended from Ecuador to central Chile for more than five thousand miles. Its capital was Cuzco established in the high Peruvian Andes. This highly advanced civilization developed a counting system used to run the empire - in particular, to build the 14,000 mile road structure and monumental architecture. Some of the algorithms believed to be used by the Inca to do computations using a *yupana*, an ancient calculating device, will be presented, as well as classroom activities for the course "Mathematics in Non-European Cultures" for non Mathematics and Science majors offered at Millersville University of Pennsylvania.

Key Words: Ethnomathematics, Inca, Yupana, Arithmetic

Resumen

El Imperio Inca – el imperio pre-Colombino más grande del continente Americano – se extendió desde Ecuador hasta Chile central por más de cinco mil millas. La capital fue Cuzco, la que fue establecida en las Alturas de los Andes Peruanos. Esta civilización altamente avanzada desarrolló un sistema contable para organizar el imperio – en particular, para construir una red de caminos de 14.000 millas y una arquitectura monumental. Algunos de los algoritmos que se creen haber sido usados por los Inca para hacer cálculos fue la "*yupana*". Este antiguo artefacto de cálculo será presentado, entre otras actividades del curso "Matemáticas en Culturas No-Europeas" para estudiantes de todas las áreas de concentración, excepto Matemáticas y Ciencia, ofrecido en la Universidad de Millersville en Pennsylvania.

Palabras Clave: Etnomatemáticas, Inca, Yupana, Aritmética

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Introduction

The Since 1993, Millersville University of Pennsylvania has offered the 3-credit course, Mathematics in Non-European Cultures (MATH 102). The course is a survey of mathematical ideas developed by non-European cultures including, but not limited to, those of Africans, Asians, and Native North, Central and South Americans. Presently, we are in the process of developing a similar course at West Chester University of Pennsylvania.

Since 2006, Dr. Ximena Catepillan has been offering a special session of the course that includes a week-long trip to Mexico's Yucatan Peninsula together with archaeologists from the Maya Exploration Center. We are planning on expanding the course to include students from West Chester University. MATH 102 is a general education course designed for students not majoring in Mathematics and Science. The students who take this course typically major in Art, Anthropology, History, Music, Social Work, and Sociology, among others.

In 2008, we traveled to the Andes to learn about the Inca civilization, culture and mathematics. We became interested in a stone artifact called y*upana*, which archaeologists claim was used to perform mathematical computations. The Inca also invented an advanced information recording device called *quipu*, which is probably one of the most well known Inca achievements.

In this article, we give a brief introduction to the Inca civilization, we continue with the spoken numbers in Quechua, the Inca language used until now by some of the native people of the Andes, and end with theories concerning computations with the *yupana*. We illustrate the uses of the *yupana* with examples.

As demonstrated by the abovementioned course at Millersville University, the material presented in this paper can be put to excellent use in a Mathematics course for liberal arts students giving such course an interdisciplinary quality.

The Inca Civilization

The most influential pre-Columbian civilizations on the American continent were the Maya and the Inca. The Maya occupied the territory of what is today Southeastern Mexico, Belize, Guatemala, El Salvador and Honduras. The Inca territory covered what is today

South Ecuador, Peru, parts of Bolivia and North and Central Chile. The civilization of the Maya, which has existed since about 1800 BC, was at its peak during the Classic Period between 250 and 800 AD. The Inca Empire existed only between ca. 1400 and 1533 AD. The Empire collapsed upon the arrival of the Spanish conquistadores led by Francisco Pizarro.

We do not have much information regarding the Inca from the Inca themselves because they did not develop a writing system. Most of the native information we have is preserved in the *quipus* which have not yet been completely deciphered. A *quipu* is a recording device consisting of colored cords containing information in the form of knots, see figure 5. Fortunately, the Spanish chroniclers left a significant amount of records about the Inca. The Inca Empire truly was an Empire – it was ruled by one emperor or king. The word "Inca" means "king" in Quechua. Later, the word "Inca" was applied as the name of the entire Empire.

Below is the chronological Inca dynastic sequence:

Manco Capac

Sinchi Roca

Lloque Yupanqui

Mayta Capac

Capac Yupanqui

Inca Roca

Yahuar Huacac

Viracocha Inca

Pachacuti Inca Yupanqui

Tupac Inca Yupanqui

Wayna Capac

Huascar and Atahualpa

An excellent interactive map of the Inca Empire can be accessed in the National Geographic site in figure 1.

Figure 1: Interactive Map of the Inca Empire

The capital of the Inca Empire was Cuzco, located at its geographic center in the high Andes at 11,600 feet altitude. To rule successfully over such vast territory of over 300,000 square miles, an efficient system of communication was necessary. This was one reason why an extensive road system of approximately 14,000 miles was developed. Also, a system of aqueducts was constructed. This proves that the Inca were advanced engineers. Engineering, in turn, requires computations – at least arithmetic.

The picture below – figure 2 - was taken in 2008 during our trip to the Andes to learn about the Inca civilization, culture and mathematics.

Figure 2: Inca girls at the plaza in Cuzco (Photo by Ximena Catepillán)

Spoken Numbers in Quechua

To begin, we list the Quechua names of the basic numbers (Urton, 1997).

One	uj
Two	iskay
Three	kinsa
Four	tawa
Five	phishqa
Six	suqta
Seven	qanchis
Eight	pusaq
Nine	jisqon
Ten	chunka
One hundred	pachaq
One thousand	waranqa

To write numbers in which all digits except the last one are one or zero we use addition, as follows:

(a) When the last digit ends in a vowel we write the word **yuq** at the end, which indicates addition.

13 chunka kinsa**yuq** 13 = 10 (chunka) + (yuk) 3 (kinsa)

114 pachaq chunka tawa**yuq** 114 = 100 (pachaq) + (yuk) 10 (chunka) + (yuk) 4 (tawa)

1004 waranqa tawa**yuq** 1004 = 1000 (waranqa) + (yuk) 4 (tawa)

1116 waranqa pachaq chunka suqta**yuq**

If a three or more digit number ends with 10 (**chunka)**, we write **chunkan** instead of chunkayuk.

1010 waranqa **chunkan**

1110 waranqa pachaq **chunkan**

(b) When the last digit ends in a consonant or semi-consonant, we write the word **niyuq**, which indicates addition.

18 chunka pusaq**niyuq** 18 = 10 (chunka) + (niyuq) 8 (pusaq)

When the last digit in an addition is **pachaq**, we write **pachaqnin** instead of pachaqniyuk, for example,

1100 waranqa **pachaqnin**

To write multiples of powers of ten we simply write:

20 = 2 tens = iskay chunka

300 = 3 hundreds = kinsa pachaq

70000 = 7 tens of thousands = qanchis chunka waranqa

1 million = one thousand of thousands = waranqa waranqa

To write arbitrary numbers:

48 = 4 tens and eight = tawa chunka pusaqniyuq = (40) (tawa chunka) + (niyuq) 8 (pusaq)

3008 = 3 thousands and eight = kinsa waranka pusaqniyuq = (3000) (kinsa waranka) + (niyuq) 8 (pusaq)

4603 = 4 thousands 6 hundreds and 3 = tawa waranqa suqta pachaq kinsayuq = 4 000 (tawa waranqa) + (yuq) 6 00(suqta pachaq) +(yuq) 3 (kinsa)

66000 = 66 thousands = suqta chunka suqtayuk waranqa = [60 (suqta chunka) +(yuk) 6 (suqta)] 1000 (waranqa)

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Exercise 1: Write the following numbers in Quechua.

a) 23

b) 124

c) 1704

d) 2222

Exercise 2: Write the following numbers in Hindu-Arabic.

a) iskay chunka ujniyuq

b) kinsa pachaq chunka pusaqniyuq

c) tawa pachaq suqta chunka jisqoniyuq

d) kinsa chunka kinsayuk waranqa kinsa pachaq kinsa chunka kinsayuq

Arithmetic - Yupana - Inca Abacus?

Yupana is a Quechua word for "herramienta para contar" (in Spanish), "tool for counting". A picture of a *Yupana* can be seen in figure 3. The artifact is a rectangular grid, usually on a piece of stone or wood, and a collection of corn kernels of different colors used to represent numbers on the grid. Corn was a fundamental staple for the Inca and it had many varieties of colors.

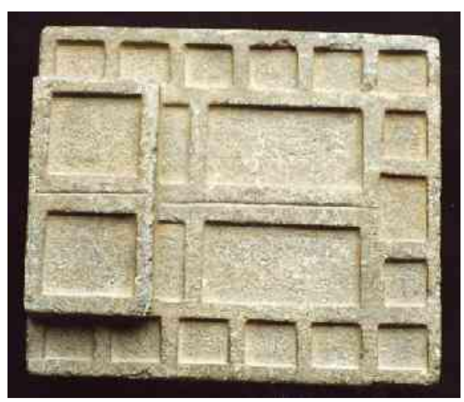

Figure 3: Yupana (Aimi & De Pasquale, 2003)

It is believed that *Yupanas* were used to make computations, the results of which were recorded on a *quipu*. The passage below from the book "*Historia Natural Moral de las Indias*" written by Father Jose de Acosta (Acosta, 1590), describes the use of a *yupana*.

Acosta, a Spanish priest who lived in Peru from 1571 to 1586, was one of the most accurate chroniclers of the sixteenth century.

> To see them use another type of quipu that employs grains of maize is a fascinating thing. For to make a very difficult calculation, to see how much each person must contribute, which an excellent accountant would have to do with pen and ink, these Indians, taking so many grains from that place and adding a certain number from this, and hesitating a hundred times, will take their grains and put one here, three there, and eight I don't know where; they will move one grain to another place, switch three from elsewhere, and their account comes out very accurately, without the slightest error; and they know much more clearly how to balance an account of what each one has to pay or give, than we could accomplish with pen and ink…. (Page 344).

To reinforce the description of Acosta, we quote the following passage from Inca Garcilaso de la Vega (Garcilaso de la Vega, 1609), who was the first Peruvian Inca/Spanish chronicler (1539 - 1616).

> De la Aritmetica supieron mucho y por admirable manera, que por nudos dados en unos hilos de diversos colores daban cuenta de todo lo que en el reino del Inca había de tributos y contribuciones por cargo y descargo. Sumaban, restaban y multiplicaban por aquellos nudos, y, para saber lo que cabia a cada pueblo, hacían las particiones con granos de maíz y piedrezuelas, de manera que les salía cierta su cuenta… (Page 167).

> Of Arithmetic they knew a lot and in an admirable manner, since by using knots on different color threads they kept track of everything that in the kingdom of the Inca had to do with taxes and dues by adding and subtracting. They added, subtracted, and multiplied using knots to keep track of each town's dues, they did the computations using kernels of corn and pebbles to obtain accurate results…(translated by Ximena Catepillán).

Felipe Guaman Poma de Ayala was the author of Nueva Crónica y Buen Gobierno, (Guaman Poma de Ayala, 1613), in which we can find a legendary drawing, see figure 5. The drawing depicts a Tawantin Suyu – an accountant in Quechua - who is holding a *quipu* and, in the lower left corner, the grid resembles a *yupana* with dots representing numerical information.

Figure 5: Drawing by Guaman Poma de Ayala (1613)

The text on the picture translates as follows: Main Accountant and Treasurer, *Tawantin Suyu khipuq kuraka*, the authority in charge of the *quipus* of the Tawantinsuyu. For brevity, in what follows we will use the word “grid” for *yupana* and “dot” for a black dot representing a corn kernel. The empty circles represent places for dots.

Let us have a look at the grid of Poma’s drawing. Henry Wassen (1931) gives the following interpretation of the following grid in figure 6:

(a) Row values, from bottom to top, are successive powers of ten.

(b) Column values, from left to right, are the values 1, 5, 15, and 30.

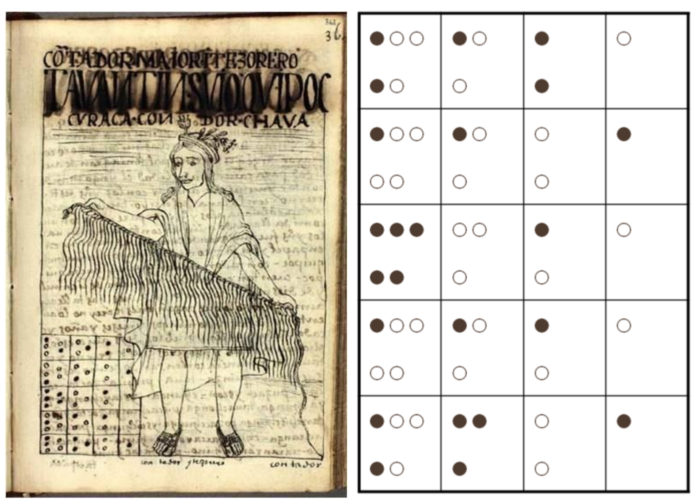

Figure 6: Yupana

The consecutive rows (from the bottom to the top) represent the following numbers:

Row 1: 2×1+ 3×5+1×30 = 47

Row 2: (1 + 5 + 15) ×10= 210

Row 3: (5 + 15) ×100 = 2000

Row 4: (1 +5+30) ×1000 = 36000

Row 5: (2+ 5 + 30) ×10000 = 370000

When we add all these numbers we get 408257.

Exercise 3: Write the beginning of Lloque Yupanqui's reign, 1260, using Wassen's method.

A much more natural, purely decimal, explanation was proposed by George Gheverghese (1990).

Row values, from bottom to top are successive powers of ten. Column values, from left to right, have the value 1.

Therefore, Poma's drawing depicts now the number

6 + 3×10 + 6×100 + 3×1000 + 5×10000 = 53636.

Exercise 4: Utilize a grid to write the year Pizarro conquered Cuzco, 1533, using the Gheverghese method.

We have 3 + 3×10 + 5×100 + 1×1000 = 1533 figure 7 depicts one of the various answers. Find others.

Figure 7: Gheverghese method

Exercise 5: Find the number represented on the grid in figure 8 using both methods.

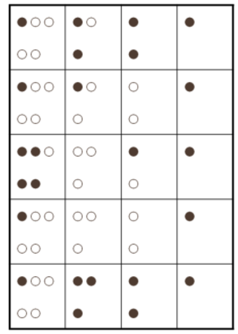

Figure 8: Use both methods

In what follows we will only use the decimal system of Gheverghese.

Now we shall discuss arithmetic operations performed using the grid. Addition, subtraction, and multiplication follow the interpretation of Carlos Radicati (Radicati 1973). When it comes to division, we will present our own interpretation. All the operations will be explained using examples. They will be performed using natural numbers {1, 2, 3,…}.

Adding numbers on the Yupana

Example 1: To add the numbers 11411 and 21003 we place them in the first two left columns of the grid. In each row, we move the dots from the first two cells into the third cell, see figure 9. In the third column, we get the answer 32414. Keep in mind that all these operations were performed manually by moving dots (kernels) from one cell of the *yupana* to another.

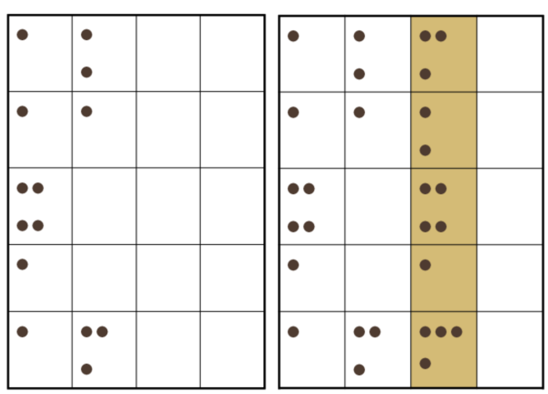

Figure 9: Adding numbers on the Yupana

Adding numbers with trades

Example 2: Let us add now 11561 and 1643. We obtain 1, 2, 11, 10, and 4 dots in the third column (from top to bottom). We have to trade since we cannot have ten or more dots in any cell. Therefore, we rewrite the third column as the fourth column with 1, 3, 2, 0, 4 dots, which means that the result is 13204. This is because every ten dots in any cell can be replaced by one dot in the cell directly above it (decimal system). Figure 10 depicts the addition.

Basic principle of trading on the yupana

A dot in a cell equals 10 dots in the cell directly below

.

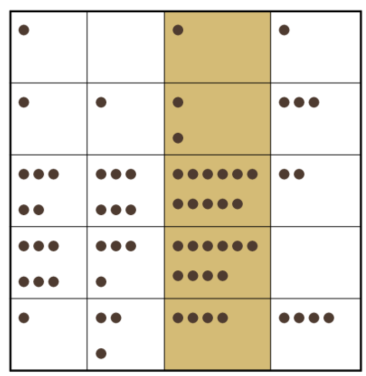

Figure 10: Example of addition

Subtracting numbers on the Yupana

Subtraction without trading is straightforward.

Example 3: To subtract 654 from 1411 we have to trade. We rewrite the first column as 0, 13, 10, 11, and then subtract obtaining the answer 757 in the last column. Figure 11 illustrates the subtraction.

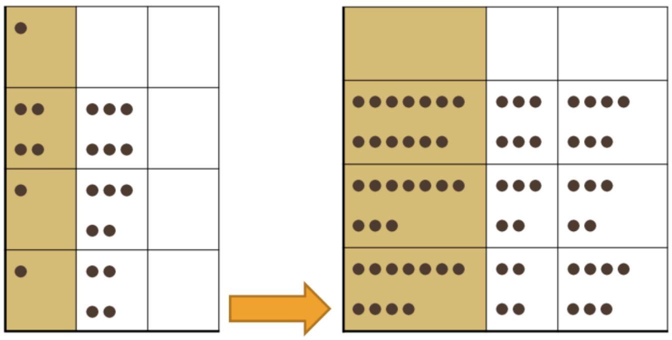

Figure 11: Example of subtraction

Multiplying numbers on the Yupana

To multiply numbers, Carlos Radicati (Radicati, 1973) was inspired by the multiplication method used by Hector Calderón (Calderón, 1966) for the vigesimal Maya number system.

Example 4: Multiply 21 by 123. Place one number (21) in the first column from the left and the second number (123) in the top row starting with second column. Multiply each digit of one number by each digit of the other number. Write the result at the intersection of the corresponding row and column. Add the resulting numbers diagonally from left bottom to right top. In the last column, we get the result 2583. The multiplication steps can be seen in figure 12.

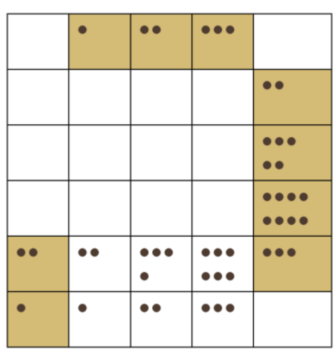

Figure 12: Example of multiplication

Multiplying numbers with trades

Example 5: To multiply 33 by 444 we will have to trade. The answer, in figure 13, is 14652.

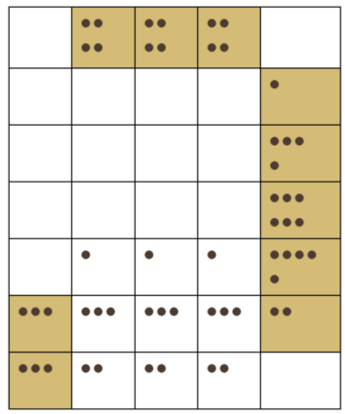

Figure 13: Example of multiplication

Dividing numbers on the Yupana

As we know, division is "reversing" multiplication. This is division without a remainder. Division with remainder will be shown later.

2×Y is multiplication, 2× *Y*Y= 6 is division, 6 ÷ 2 = Y.

In the division a ÷ b = c, a is called the dividend, b the divisor, c the quotient.

Hence dividend = divisor × quotient.

Example 6: (No remainder, no trading). 2583 ÷ 21 Let us examine the multiplication we had in example 4: 21×123 = 2583.

Step 1: Set-up

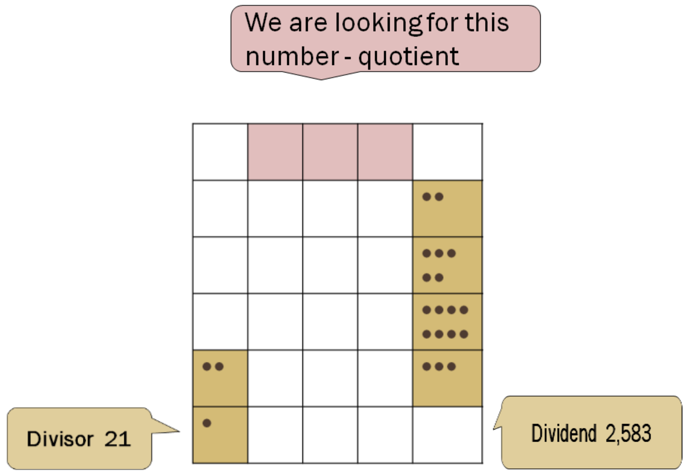

Figure 14: Set-up

Look at the divisor, in figure 14, and try to fit it into the number formed by consecutive digits of the dividend starting from the top: 21 does not go into 2, therefore 21 into 25 goes one time. Place 1 in the top row second column as seen in figure 15.

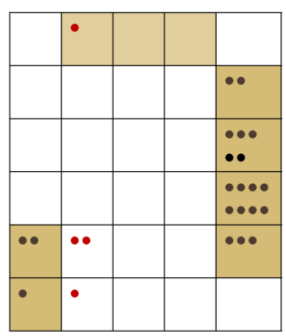

Figure 15: First step

Step 2: Multiply 1×21 to obtain 21, on a side we compute the difference 25 – 21 = 4. Now look at the next digit from the top of the dividend, 8, and form the number 48, see figure 16.

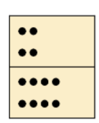

Figure 16: The number 48

21 goes into 48 two times, we place 2 in the top row, third column, figure 17, 2×21= 42.

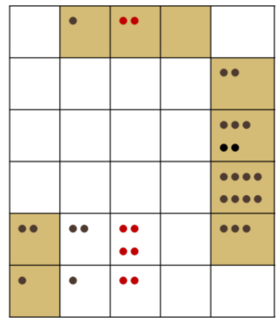

Figure 17: Step 2

Step 3: Again, on a side, compute the difference 48 – 42 = 6. Take the last digit of the dividend, 3, and form the number 63. 21 goes into 63 three times. Place 3 in the top row, fourth column. Multiply 3×21 to get 63.

In this manner, we have recovered exactly the multiplication grid of Example 4. The top row in figure 18 displays the quotient. Multiplication checks that division is correct.

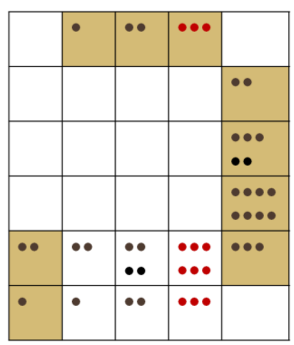

Figure 18: Step 3

Example 7: (No remainder, trading). Divide 1378 by 26.

Step 1: Set-up in figure 19.

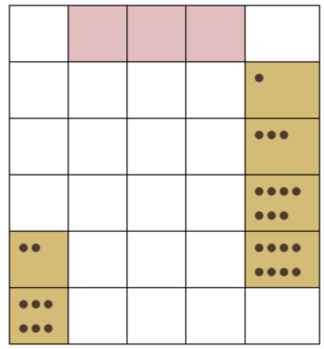

Figure 19: Set-up

26 goes into 137 five times.

Step 2: Multiply 5×26 to get 130, subtract 137 – 130 = 7. Get the next (last) digit of the dividend, 8. In figure 20 we represent how 26 goes into 78 three times.

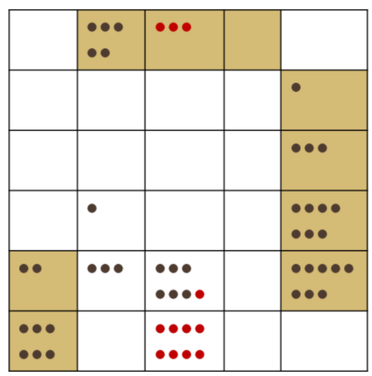

Figure 20: Step 2

Step 3:

Multiply 3×26 to get 3×2 = 6 (tens) plus 3×6 = 18, which means 1 more dot on the row of tens and 8 dots in the row of units (trade). Hence, the quotient is 53.

Division with remainder

Suppose we want to divide 3 into 16 in natural numbers. We get quotient 5 and remainder 1. That is 16 = 3×5 + 1. In general,

dividend = divisor × **quotient + remainder**

Example 8: (Trades and remainder). Consider 1381 ÷ 26. We just start as in Example 7.

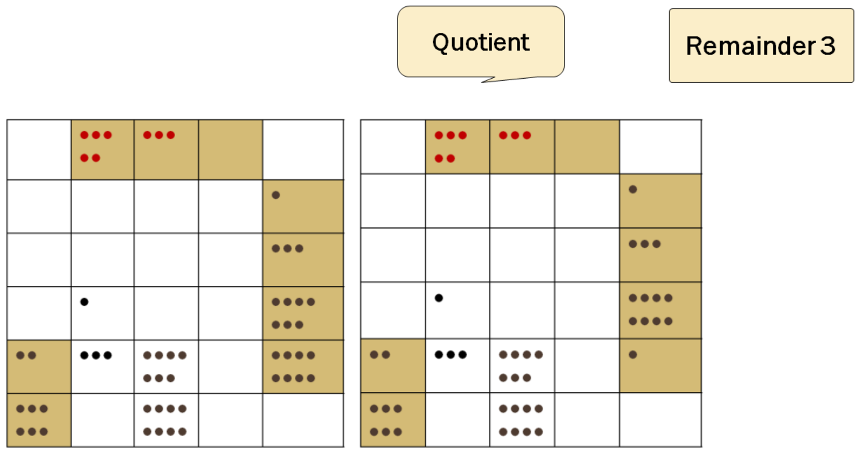

Figure 21: Division with remainder

Notice that the left hand side grid in figure 21 is the grid in figure 20. To find the remainder we have to subtract 1378, the last column in the left hand side grid in figure 21 from 1381, the last column of the right hand side grid in the same figure. Therefore, the remainder is 3. This means: dividend = divisor × quotient + remainder, which in this example is $1381 = 26\times53 + 3$

Conclusion

The material presented in this paper can be of great use in a mathematical ideas course for non-Mathematics/Science majors. Not only does it contain motivating Mathematics, it also gives "real life" historical explanations as to why Mathematics had to be developed by this ancient civilization. On the practical side, students could make a real *yupana* and, using beans, perform the operations described in this paper. It is not difficult to see that performing operations on the *yupana* is much easier and faster than traditional style on paper – see quote by father Jose de Acosta on page 8. Studying the Mathematics of the Inca also gives us the opportunity to discuss the history and culture of an outstanding ancient civilization. In summary, the arithmetic on the *yupana* provides a wonderful educational tool that can be used for both university and high school students.

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